

Operators having selfadjoint squares

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ABSTRACT. The main goal of this paper is to show that a (not necessarily densely defined or closed) symmetric operator A acting on a real or complex Hilbert space is selfadjoint exactly when $I + A^2$ is a full range operator.

1. Introduction

If T is a densely defined closed operator between two Hilbert spaces, \mathfrak{H} and \mathfrak{K} , a classical theorem due to John von Neumann [3] states that $I + T^*T$ is selfadjoint operator with full range. As an immediate consequence of that result one obtains also that the square of a selfadjoint operator, say A , is selfadjoint as well, furthermore, that $I + A^2$ is surjective. If the underlying Hilbert space \mathfrak{H} is complex, by employing the classical theory of deficiency indices, also due to von Neumann [2], we conclude that the converse of the latter statement is also true. Precisely, if A is a densely defined symmetric operator in a complex Hilbert space \mathfrak{H} such that $I + A^2$ is surjective, then the original operator A must be selfadjoint. Indeed, according to the factorizations

$$(1) \quad A^2 + I = (A + iI)(A - iI) = (A - iI)(A + iI),$$

it is seen readily that both $A \pm iI$ must be onto, and therefore that A is selfadjoint.

Factorization (1) cannot be used, of course, when the underlying Hilbert space \mathfrak{H} is real. Furthermore, if the symmetric operator is not densely defined, the theory of deficiency indices is again unapplicable, even if \mathfrak{H} is complex.

The main purpose of this note is to prove that the following characterization of self-adjointness holds, be the underlying Hilbert space real or complex: a symmetric operator A on a (real or complex) Hilbert space is selfadjoint if and only if $I + A^2$ is surjective. Observe also that the symmetric operator under consideration is not assumed to be densely defined a priori. On the contrary, densely definedness is also a direct consequence of our other assumptions.

2. Operators having selfadjoint squares

Recall that an operator A defined in a Hilbert space \mathfrak{H} is said to be symmetric if

$$(Ax | y) = (x | Ay), \quad x, y \in \text{dom } A,$$

and skew-symmetric if

$$(Ax | y) = -(x | Ay), \quad x, y \in \text{dom } A.$$

2010 *Mathematics Subject Classification.* Primary 47B25, 47B65.

Key words and phrases. Essentially selfadjoint operators, symmetric operators, skew-adjoint operators, skew-symmetric operators.

If A is densely defined in addition then the symmetry (resp., skew-symmetry) of A means that $A \subseteq A^*$ (resp., $A \subseteq -A^*$). Furthermore, a densely defined operator A is said to be selfadjoint (resp., skew-adjoint) if $A = A^*$ (resp., $A = -A^*$). Note also immediately that each selfadjoint (resp., skew-adjoint) operator is closed.

Our first result is a characterization of the skew-adjointness of an operator in terms of its square:

Theorem 2.1. *Let \mathfrak{H} be real or complex Hilbert space and $A : \mathfrak{H} \rightarrow \mathfrak{H}$ a skew-symmetric linear operator, whose domain $\text{dom } A$ is not assumed to be dense. The following statements are equivalent:*

- (i) A is densely defined and skew-adjoint;
- (ii) $-A^2$ is a (positive) selfadjoint operator;
- (iii) $I - A^2$ is a full range operator, i.e. $\text{ran}(I - A^2) = \mathfrak{H}$.

Proof. If A is skew-adjoint, then clearly, A is closed, and the following identity

$$-A^2 = A^*A$$

shows statement (ii), thanks to von Neumann's classical theorem. By assuming (ii), the operator $I - A^2$ is positive and selfadjoint, and bounded below (by one), therefore its range is dense, and closed in \mathfrak{H} , that is $\text{ran}(I - A^2) = \mathfrak{H}$. Assume finally that the symmetric operator $I - A^2$ is of full range. Then it is densely defined and positive selfadjoint, as we see at once. First, $\text{dom}(I - A^2)$ is dense, for if y is from $\{\text{dom}(I - A^2)\}^\perp = \{\text{dom } A^2\}^\perp$, then one takes into account that $y = (I - A^2)z$ for some $z \in \text{dom } A^2$. We have at once for each x from $\text{dom}(I - A^2)$ that

$$0 = (x | (I - A^2)z) = ((I - A^2)x | z).$$

Therefore, z belongs to $\{\text{ran}(I - A^2)\}^\perp = \{0\}$ by assumption, thus $y = 0$, as claimed.

One more consequence is that A is densely defined skew-symmetric operator, thus fulfilling the following identity:

$$A \subset -A^*.$$

To prove statement (i) one checks only that $\text{dom } A^* \subseteq \text{dom } A$. Let now $y \in \text{dom } A^*$ and take some $z \in \text{dom } A^2$ by assumption so that

$$y - A^*y = (I - A^2)z = (I + A)(I - A)z.$$

Then we have, since $I + A \subset I - A^*$, that

$$\begin{aligned} (y - (I - A)z) &\in \ker(I - A^*) = \ker(I - A)^* = \{\text{ran}(I - A)\}^\perp \\ &\subset \{\text{ran}(I - A^2)\}^\perp = \{0\}. \end{aligned}$$

This means just that $y = (I - A)z \in \text{dom } A$, as it is claimed. □

The main result of our paper is the following statement:

Theorem 2.2. *Let \mathfrak{H} be real or complex Hilbert space and $A : \mathfrak{H} \rightarrow \mathfrak{H}$ a symmetric operator whose domain is not assumed to be dense subspace in \mathfrak{H} . The following assertions are equivalent:*

- (i) A is densely defined and selfadjoint operator;
- (ii) A^2 is a positive selfadjoint operator;
- (iii) $I + A^2$ is a full range operator, i.e. $\text{ran}(I + A^2) = \mathfrak{H}$.

Proof. If A is assumed to be selfadjoint, then $A^2 = A^*A$ is positive selfadjoint operator in virtue of von Neumann's classical theorem. Therefore, (i) implies (ii). Statement (ii) also clearly implies (iii) as $(I + A^2)$ is positive selfadjoint and bounded below (by one) operator whose range and closed as well, therefore is the whole space \mathfrak{H} . It remains to prove implication (i) \Rightarrow (ii). First of all, A^2 is densely defined: for if y is from $\{\text{dom } A^2\}^\perp$, then, since $y = (I + A^2)z$ for some z from $\text{dom } A^2$, and at the same time for each x from $\text{dom } A^2$

$$0 = (x | y) = (x | (I + A^2)z) = ((I + A^2)x | z)$$

holds true. This means, of course, that z is from $\{\text{ran}(I + A^2)\}^\perp = \{0\}$, and therefore that $y = 0$, indeed. One more consequence is that $\text{dom } A$ is dense as well, and thus

$$A \subset A^*,$$

by our assumption on the symmetricity of A .

The last step in to check that A is selfadjoint is that $\text{dom } A^* \subseteq \text{dom } A$ as follows. Take $z \in \text{dom } A^*$, then for some x and y from $\text{dom } A^2$ we have that

$$A^*z = (I + A^2)x \quad \text{and} \quad -z = (I + A^2)y.$$

This means at the same time that

$$\begin{cases} -z = A(x + Ay) - (Ax - y), \\ A^*z = A(Ax - y) + (x + Ay), \end{cases}$$

and consequently, since $Ax - y \in \text{dom } A \subseteq \text{dom } A^*$, that $(z - (Ax - y)) \in \text{dom } A^*$ and

$$A^*(z - (Ax - y)) = A^*z - A^*(Ax - y) = A^*z - A(Ax - y) = x + Ay,$$

and as well that

$$0 = -A^*z + A^*z = A^*A(Ax + y) + (x + Ay).$$

As a consequence we finally have that

$$\begin{aligned} 0 &= (A^*A(Ax + y) | Ax + y) + (x + Ay | x + Ay) \\ &= \|A(Ax + y)\|^2 + \|x + Ay\|^2. \end{aligned}$$

Therefore $x + Ay = 0$, so that $z = Ax - y \in \text{dom } A$, indeed. The proof is complete. \square

Corollary 2.3. *Let (X, \mathfrak{M}, μ) be a measure space and f be any real valued measurable function of X . The multiplication operator A by f on the (real or complex) Hilbert space $\mathcal{L}^2(X, \mathfrak{M}, \mu)$ with maximal domain,*

$$\text{dom } A = \{g \in \mathcal{L}^2(X, \mathfrak{M}, \mu) \mid f \cdot g \in \mathcal{L}^2(X, \mathfrak{M}, \mu)\},$$

is selfadjoint.

Proof. It is readily seen that A is a symmetric operator. For a given $g \in \mathcal{L}^2(X, \mathfrak{M}, \mu)$, one obtains at once that $h = \frac{g}{1 + f^2}$ belongs to $\text{dom } A^2$ so that $(I + A^2)h = g$. That means precisely that $I + A^2$ is of full range, and therefore, in account of Theorem 2.2, that A is selfadjoint. \square

Theorem 2.4. *Let \mathfrak{H} be a real or complex Hilbert space, and $A : \mathfrak{H} \rightarrow \mathfrak{H}$ be a positive symmetric operator, not assumed to be densely defined. The following statements are equivalent:*

- (i) A is selfadjoint;
- (ii) $I + A$ is of full range, i.e. $\text{ran}(I + A) = \mathfrak{H}$.

Proof. It is clear that (i) implies (ii): $I + A$ is bounded below (by one) closed operator, therefore its range is dense and closed, i.e. $\text{ran}(I + A) = \mathfrak{H}$. Conversely, $I + A$ to be a full range operator. First of all A is densely defined: for if $y \in \{\text{dom } A\}^\perp$ then, since $y = (I + A)z$ for some $z \in \text{dom } A$ and then for each x from $\text{dom } A$ we have that

$$0 = (x | y) = (x | (I + A)z) = ((I + A)x | z).$$

Therefore, $z \in \{\text{ran}(I + A)\}^\perp = \{0\}$, i.e. $z = 0$ and then $y = 0$ as claimed.

Next we have at once that

$$(I + A) \subset (I + A)^* = (I + A^*),$$

so that $A^* = A$ is the same as $(I + A)^* = I + A$. If $y \in \text{dom } A^*$ then we see that for some $z \in \text{dom } A$

$$y + A^*y = (I + A)z = (I + A^*)z,$$

and therefore

$$(y - z) \in \ker(I + A^*) = \{\text{ran}(I + A)\}^\perp = \{0\}.$$

Consequently, $y = z \in \text{dom } A$, as it is claimed. \square

Corollary 2.5. *Let \mathfrak{H} and \mathfrak{K} be real or complex Hilbert spaces, $T : \mathfrak{H} \rightarrow \mathfrak{K}$ be densely defined linear operator. Then T^*T is positive selfadjoint if and only if $\text{ran}(I + T^*T) = \mathfrak{H}$. If T is closed, then T^*T is positive selfadjoint operator on \mathfrak{H} .*

Proof. We should only check that if T is closed then $\text{ran}(I + T^*T) = \mathfrak{H}$. Of course, this is the case when the two closed subspaces are orthogonal complements on $\mathfrak{H} \times \mathfrak{K}$:

$$\{(x, Tx) \mid x \in \text{dom } T\} \quad \text{and} \quad \{(-T^*z, z) \mid z \in \text{dom } T^*\}.$$

Therefore, for each $y \in \mathfrak{H}$ we find $x \in \text{dom } T$ and $z \in \text{dom } T^*$ such that

$$y = x - T^*z \quad \text{and} \quad 0 = Tx + z.$$

Consequently, $-z = Tx \in \text{dom } T^*$ and $-T^*z = T^*Tx$ so that

$$y = x + T^*Tx \in \text{ran}(I + T^*T),$$

as desired. \square

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